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भारतीय प्रौद्योगिकी संस्थान गोवा

CHENNAI

MATHEMATICAL

INSTITUTE

Deciding Conjugacy of a Rational Relation

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Joint work with **C. Aiswarya** (CMI) and **Amaldev Manuel** (IIT Goa)

Highlights of Logic, Games and Automata 2024

Rational Relation

- Rational relation - relation accepted by a finite state transducer.

  - expressed using rational expression over pairs of words.

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built out of pair of words using the operations union, concatenation and Kleene star.

$$(u, v) \cdot (x, y) = (ux, vy)$$

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- Example:

$$\{(u, v) \mid u, v \in \{a, b\}^*, \text{ and } v \text{ is obtained from } u \text{ by duplicating } a\text{'s}\}$$

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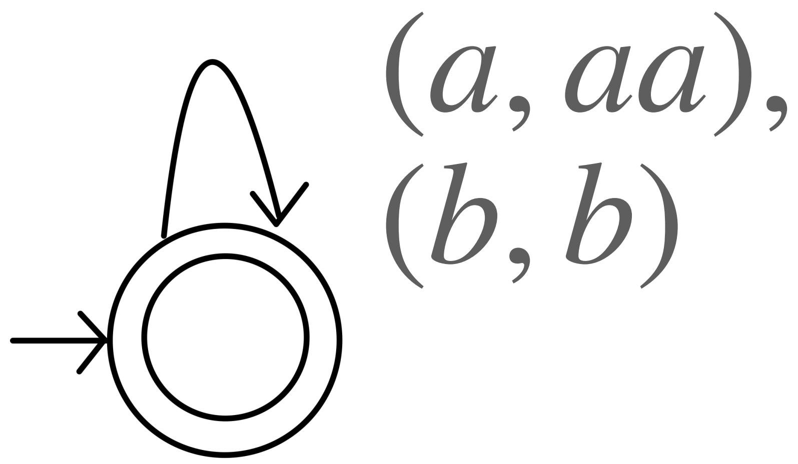
$$((a, aa) + (b, b))^*$$



Instance of Post Correspondence Problem

- Given a collection of dominos tiles,

$$\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \}$$

- PCP asks does there a sequence of dominos $i_1, i_2, \ldots, i_l$ such that

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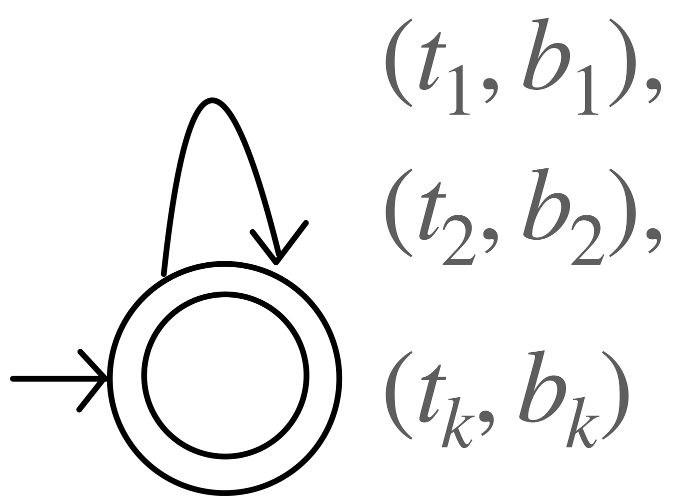
- Modelled using a rational relation

$$R = ((t_1, b_1) + (t_2, b_2) + \cdots + (t_k, b_k))^*$$



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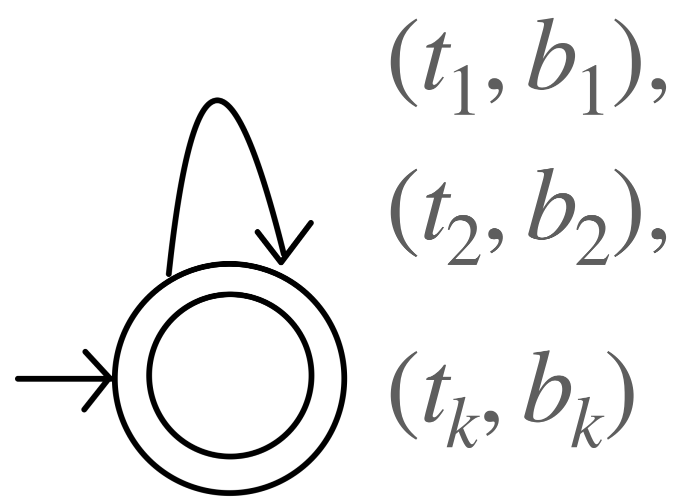

- Whether R contains an identical pair?

Instance of Post Correspondence Problem

(Undecidable [Post'46])

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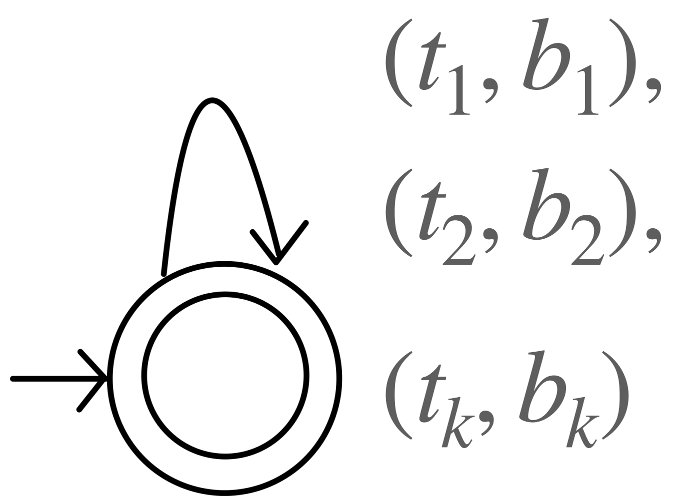

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Undecidable

We study conjugacy of rational relations

Conjugate Words

Two words u and v are conjugates

- if there exist words x and y such that $\boxed{u = xy \text{ and } v = yx}$

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l i s t e n	h e l l o	w a s
e n l i s t	h e l l o	s a w

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listen	hello	was
enlist	hello	saw

- if there exists a word z (witness) such that $uz = zv$ [Lyndon-Schützenberger'62]

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Does a rational relation contain a conjugate pair?

Instance of Conjugate PCP

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Does a rational relation contain **only** conjugate pairs?

Instance of for-all variant of PCP

- Given a collection of dominos tiles,

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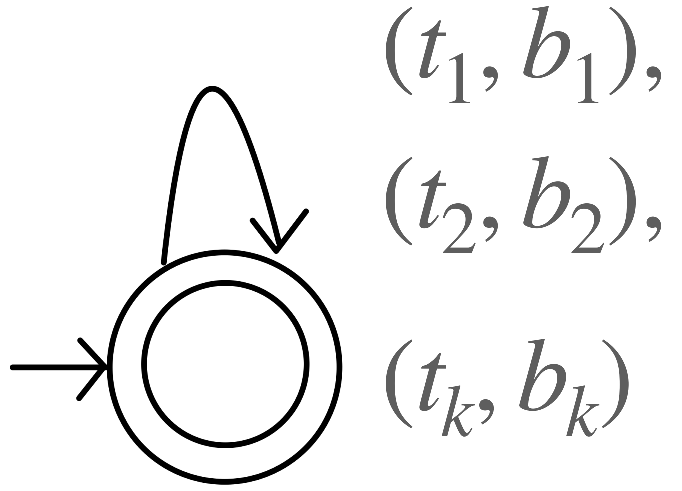

- Forall variant of PCP asks whether for every sequence of dominos $i_1, i_2, \ldots, i_l$

$$t_{i_1} t_{i_2} \cdots t_{i_l} = b_{i_1} b_{i_2} \cdots b_{i_l}$$

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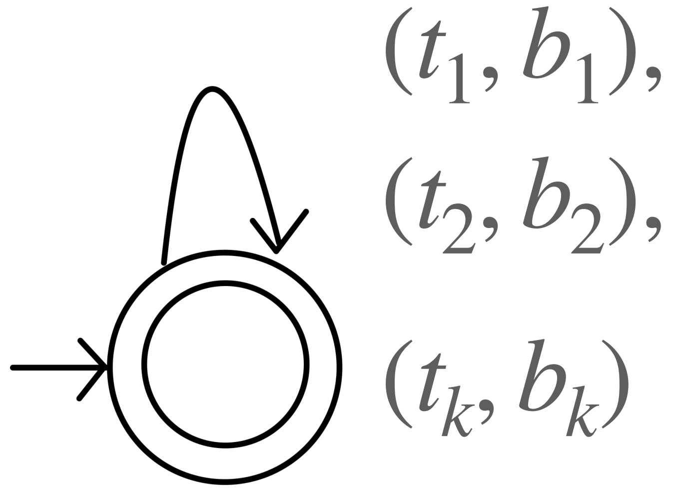

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- Whether R contains only identical pair?

Yes if $t_1 = b_1$, $t_2 = b_2$, ..., $t_k = b_k$

No otherwise

Instance of for-all variant of Conjugate PCP

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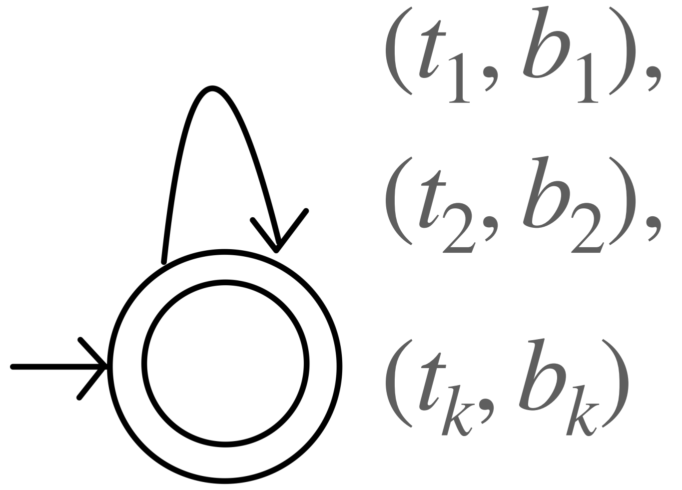

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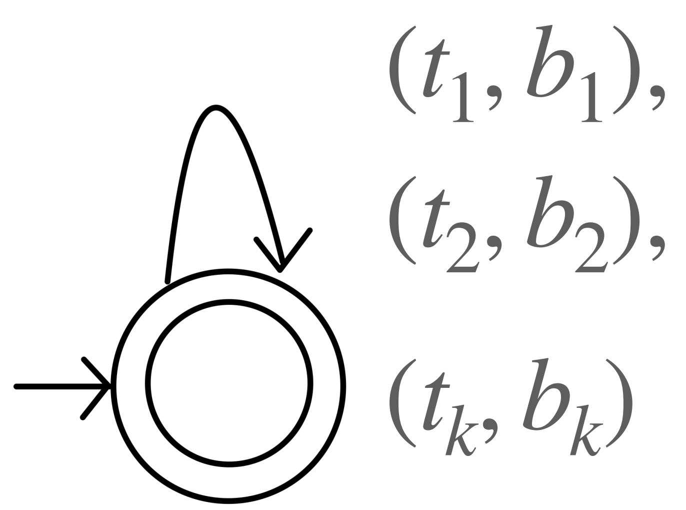
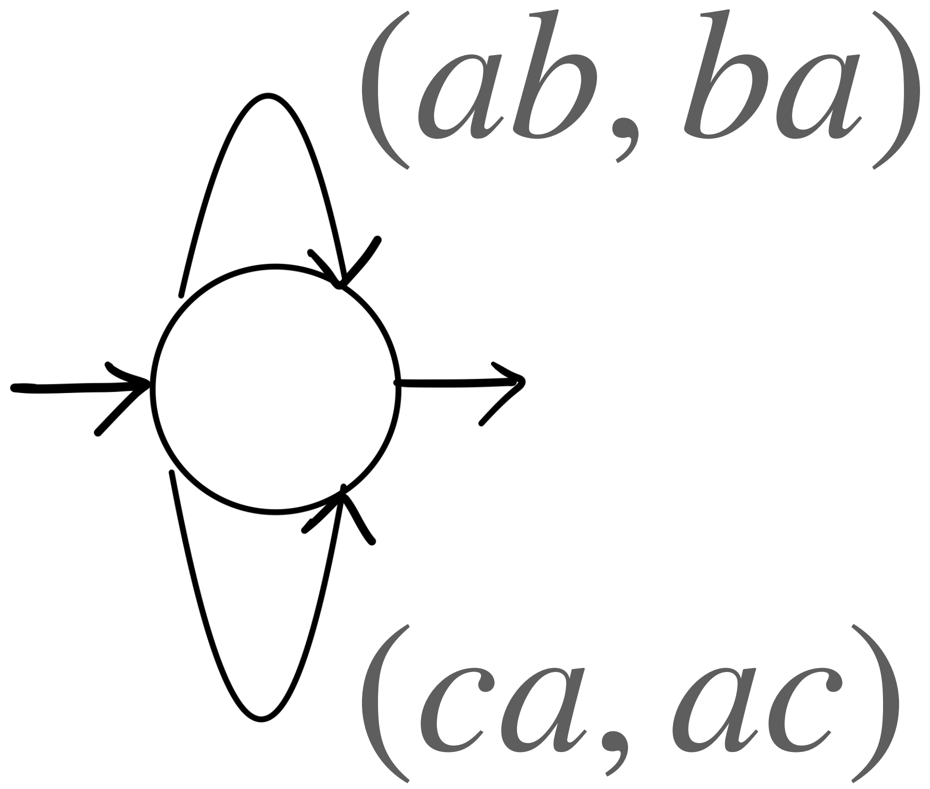


- Whether R contains only conjugate pair?

Is it enough if $t_1 \circlearrowleft b_1, t_2 \circlearrowleft b_2, \ldots, t_k \circlearrowleft b_k$?

More on Conjugates



$(abca, baac)$ is not conjugate

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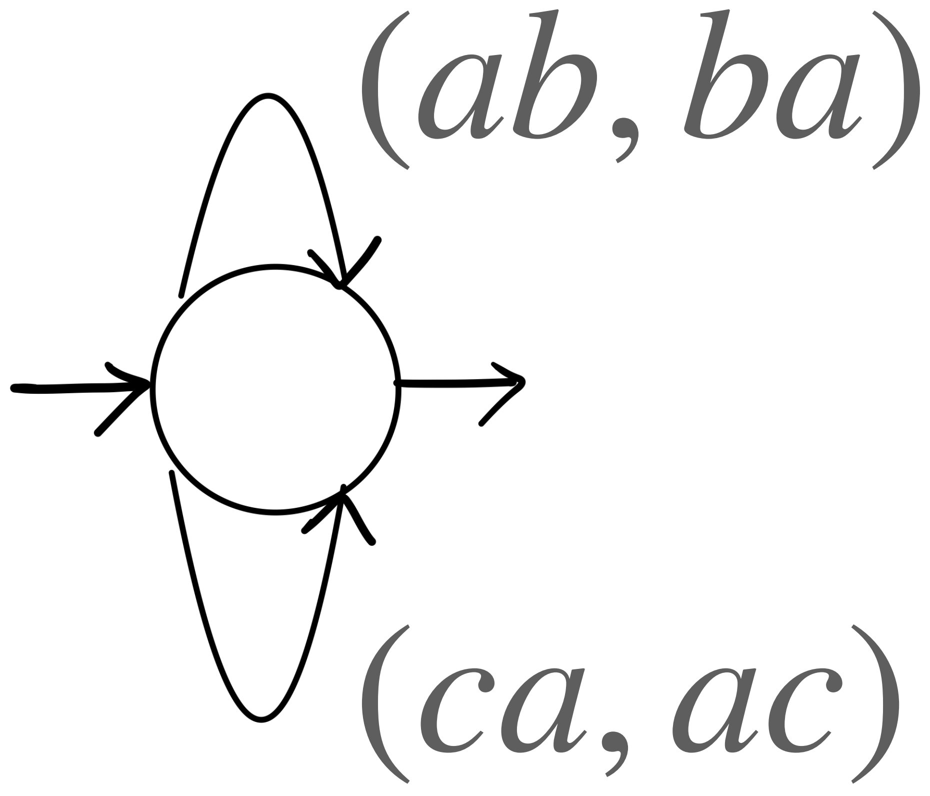

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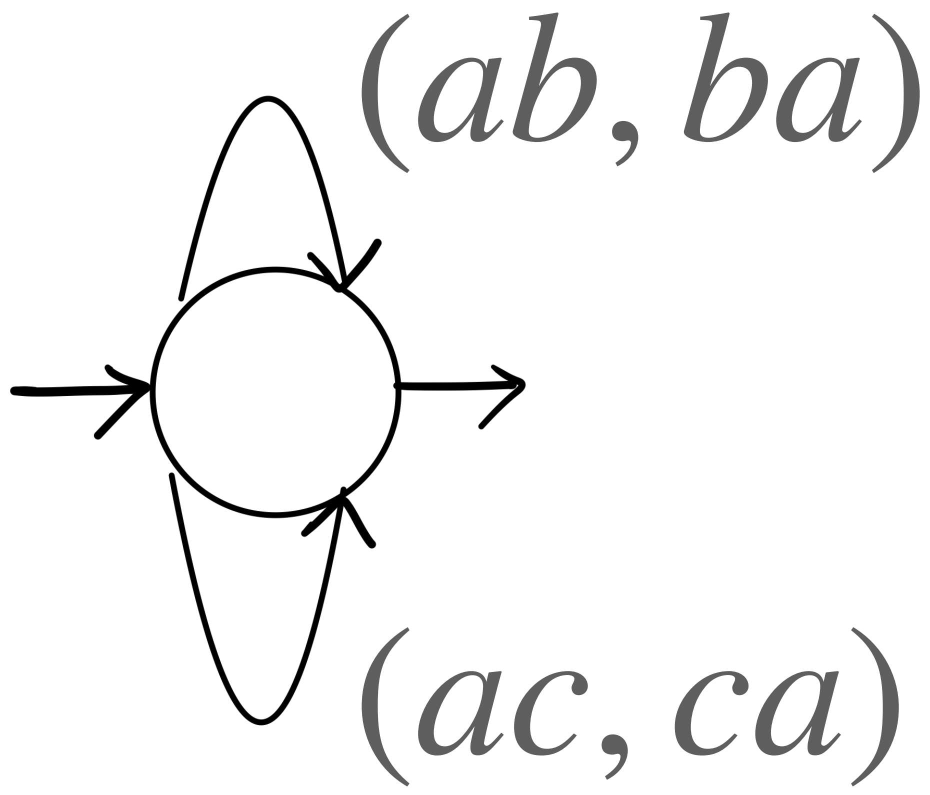

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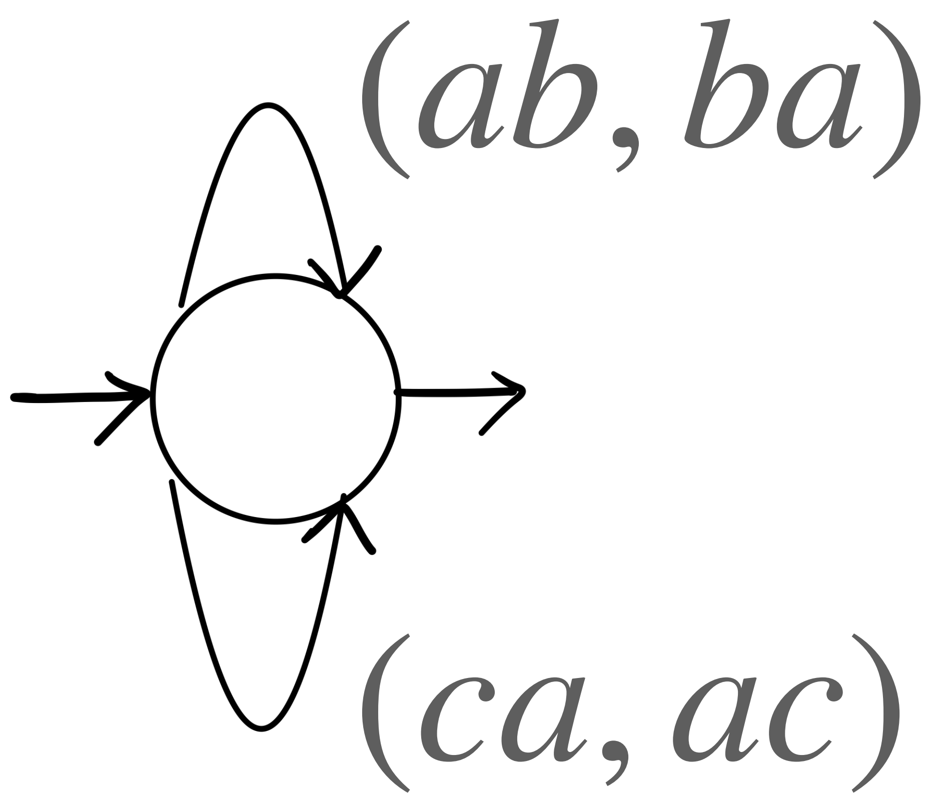

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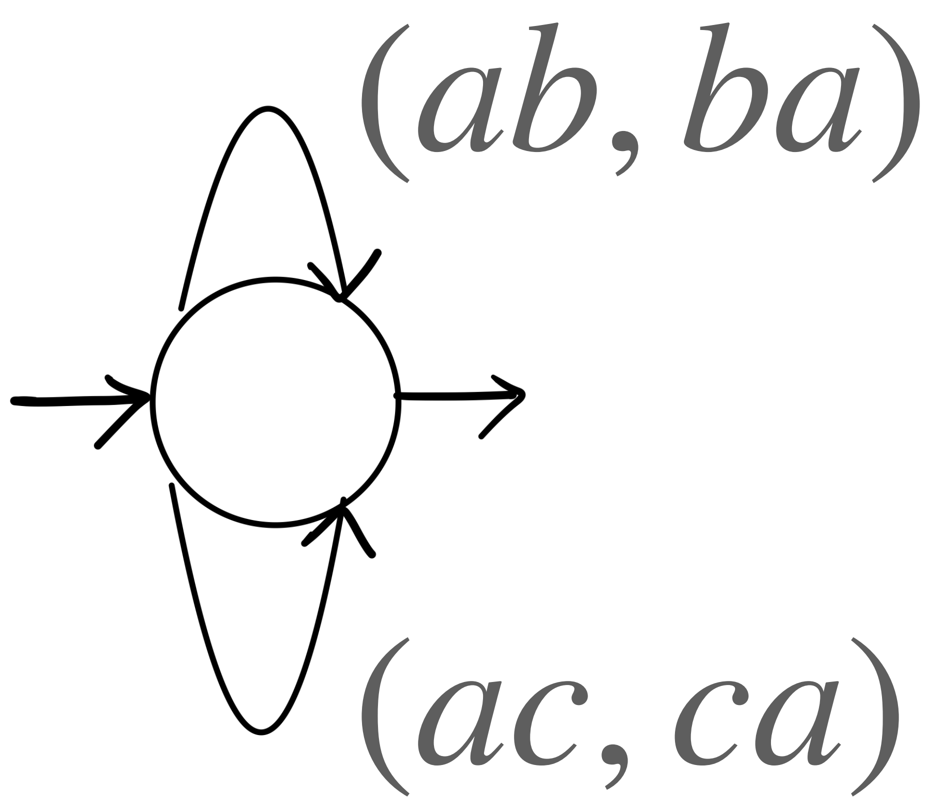

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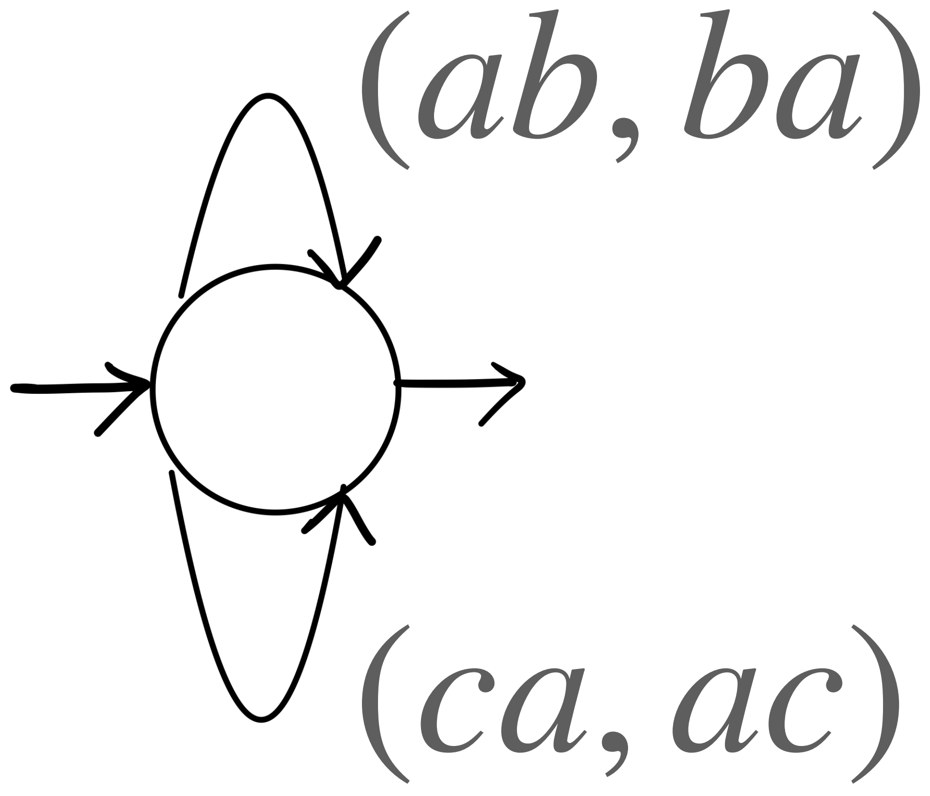

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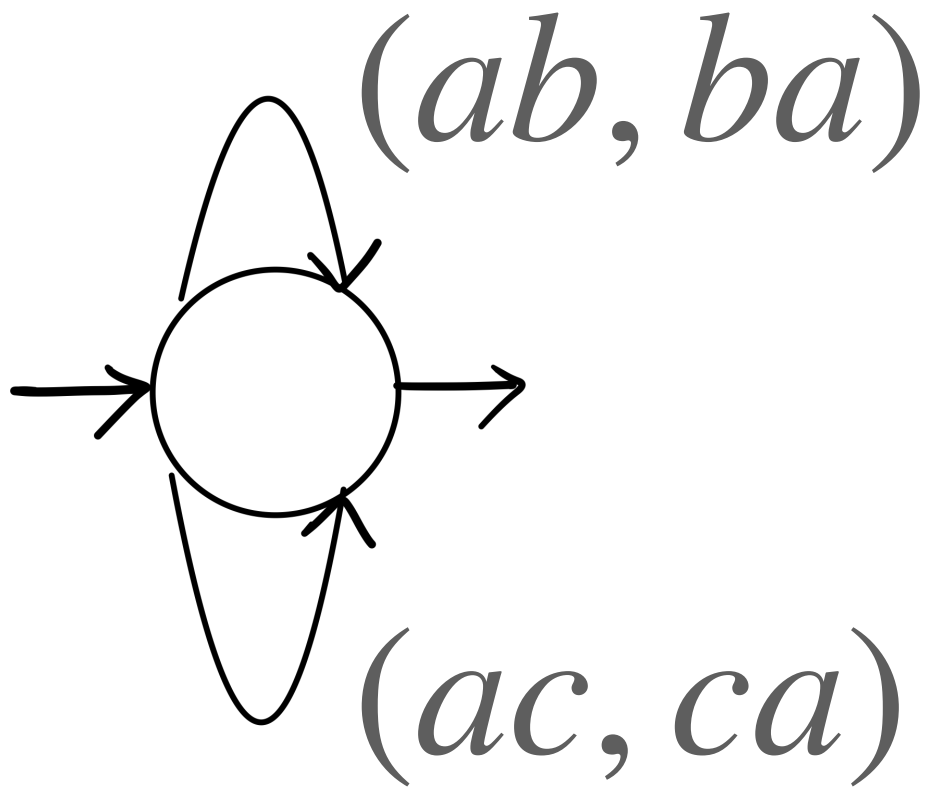

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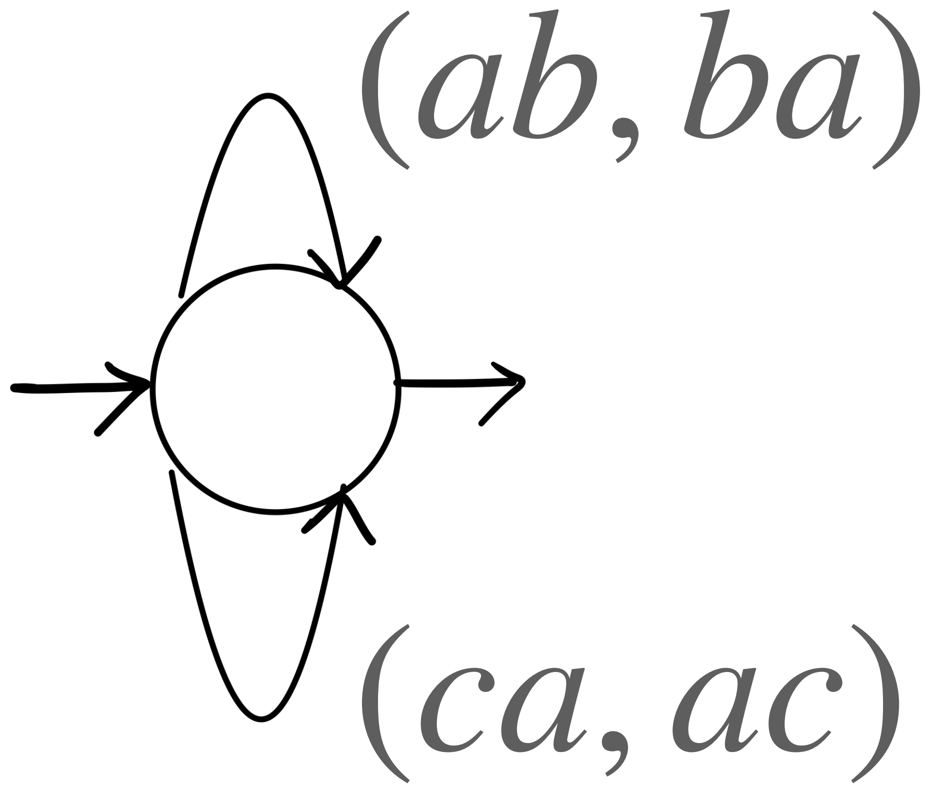

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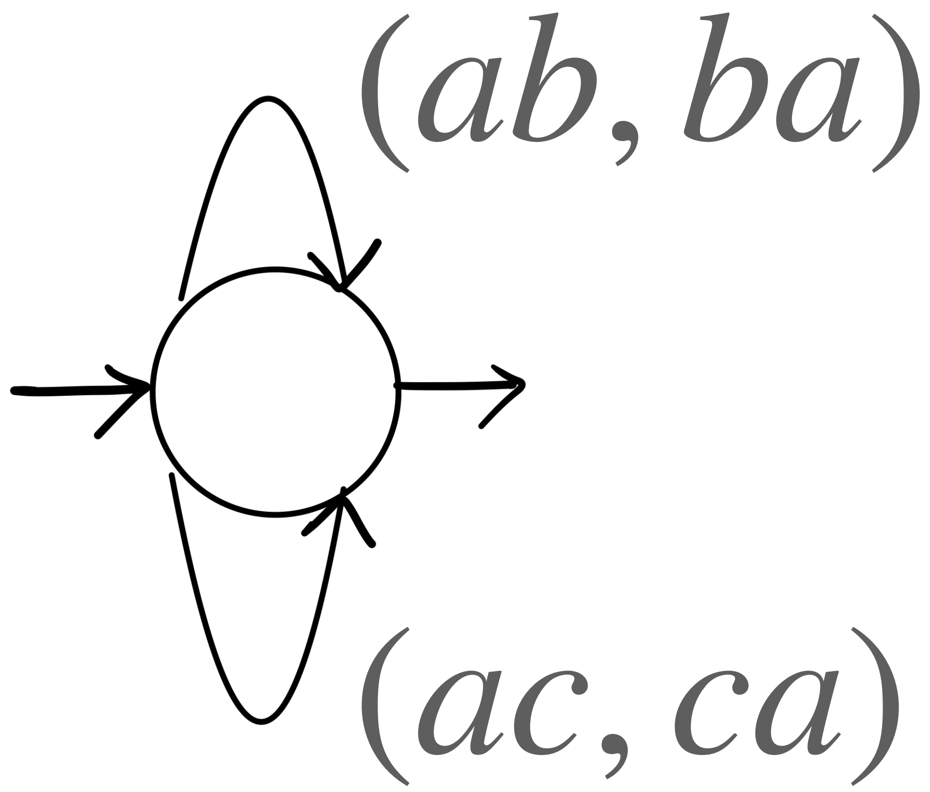

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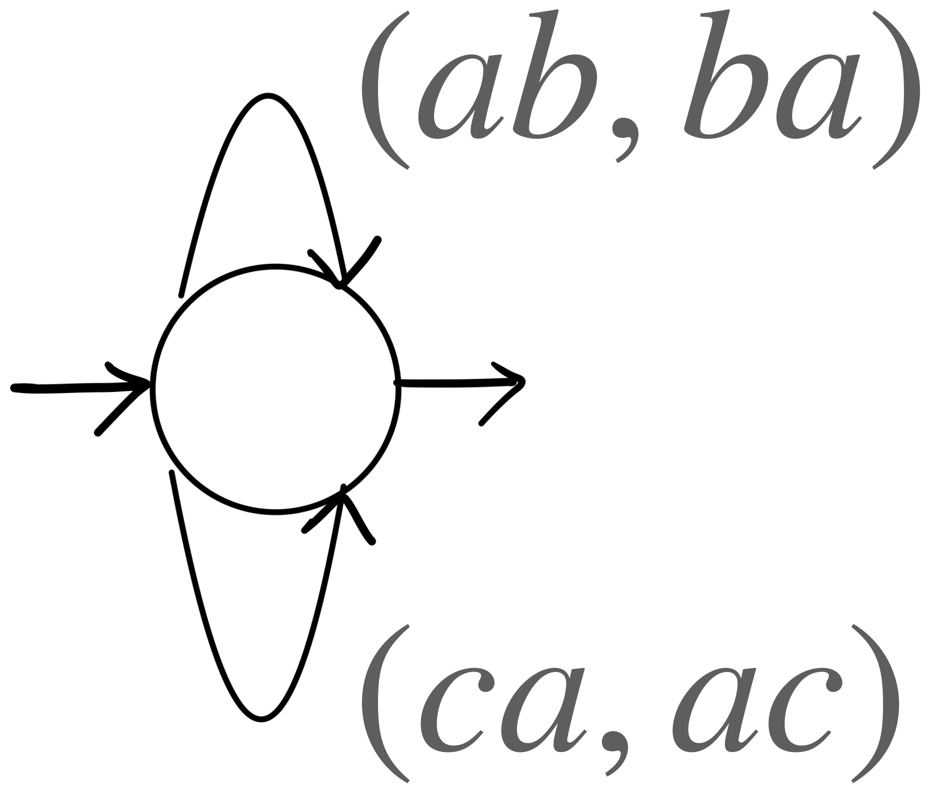

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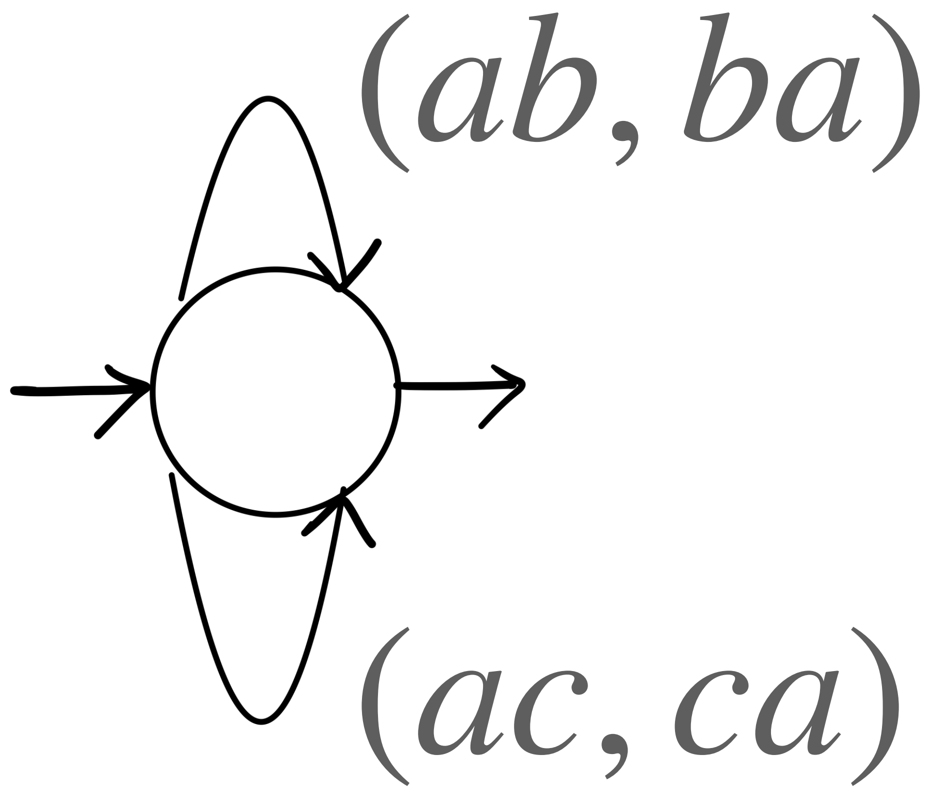

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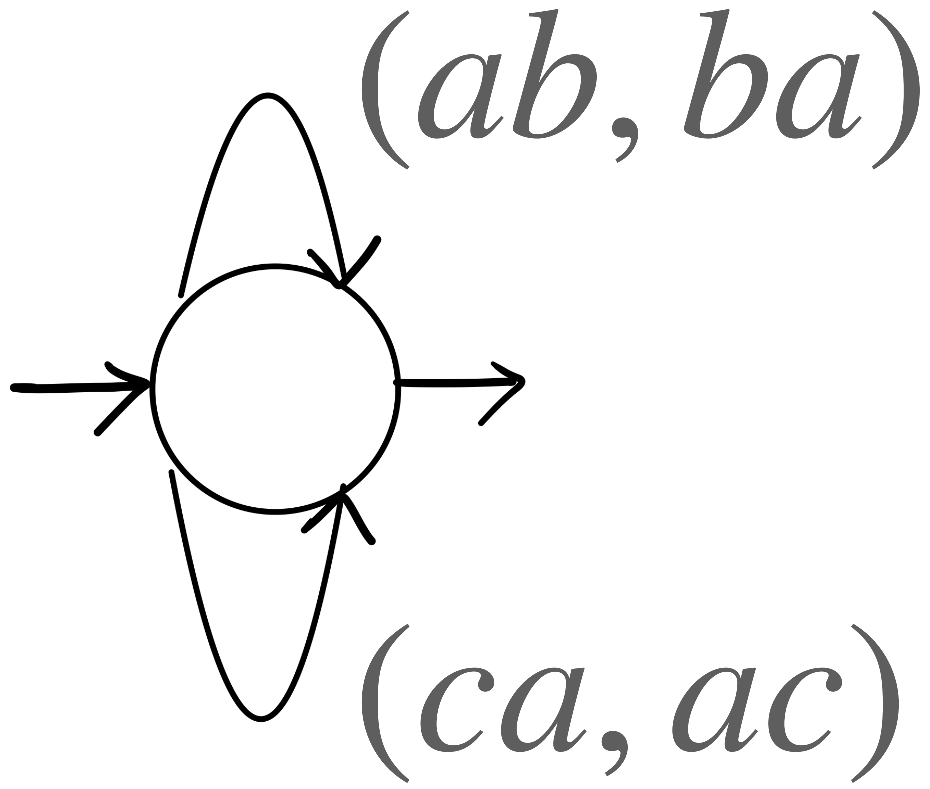

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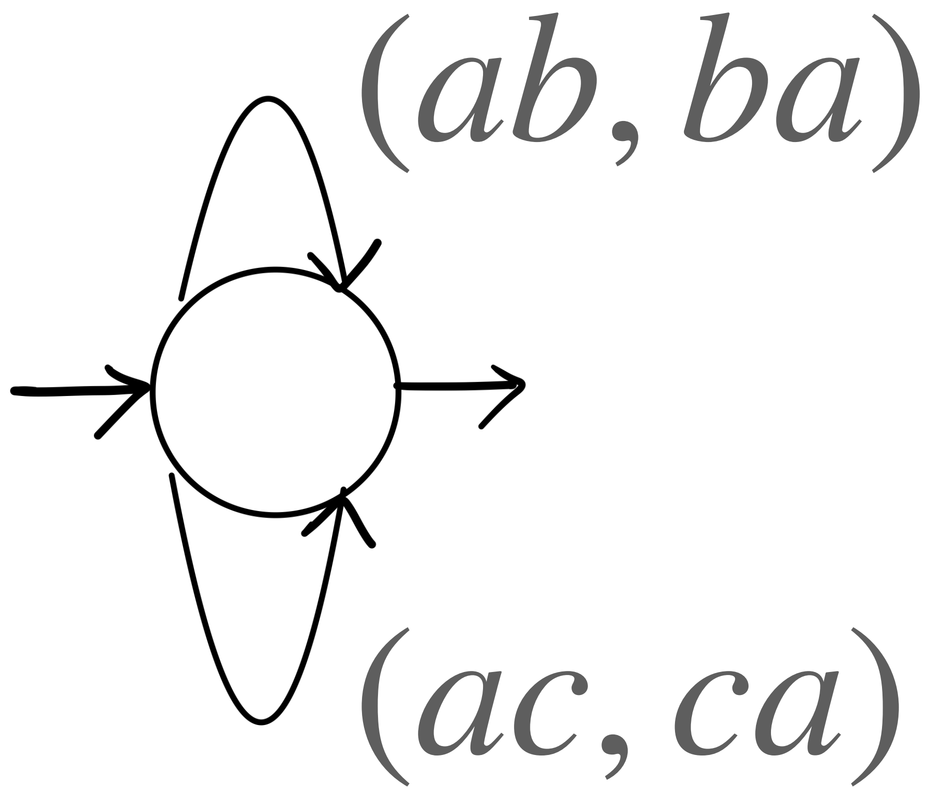


Any combination of pairs is conjugate

$$a\,b\cdots a\,b\,a\,c\cdots a\,c$$
$$b\,a\cdots b\,a\,c\,a\cdots c\,a$$

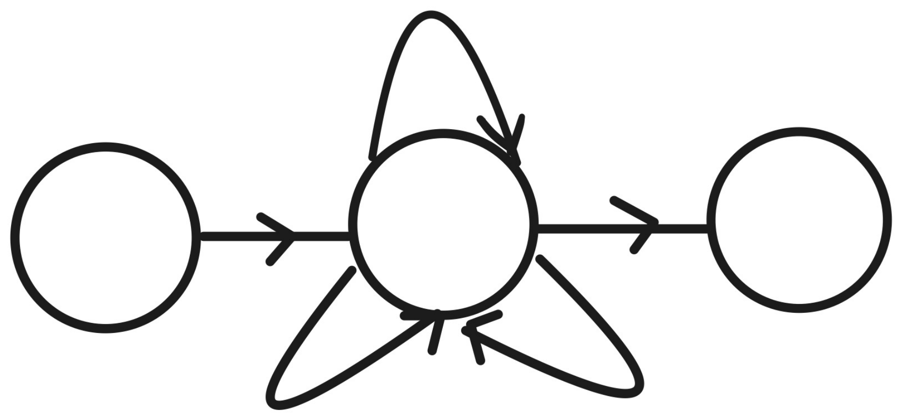
$$\boxed{uz = zv}$$

Conjugacy of Kleene Star

G — set of pairs of words

G^* — consist of pairs obtained by point wise concatenation of some pairs in G

When is every pair in G^* conjugate?



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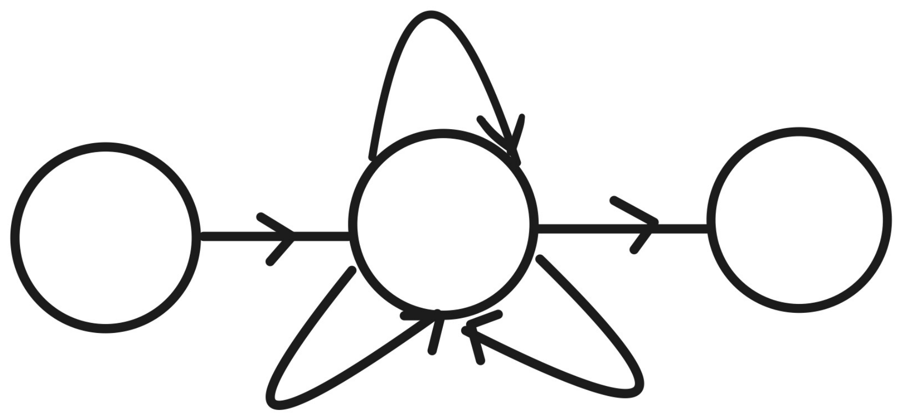

Theorem: The set G^* is conjugate iff G has a common witness

Conjugacy of Sumfree Set

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Theorem: The set $(u_0, v_0)G_1^*(u_2, v_2)G_2^* \cdots G_k^*(u_k, v_k)$ is conjugate iff it has a common witness

- This does not generalise to arbitrary sets.
- Example : $(ab, ba)^* + (ba, ab)^*$ is an infinite conjugate set but does not have a common witness.

Conjugacy of a Rational Relation

- A *set of pairs (or a relation) is conjugate* if each pair in the set is conjugate.

Theorem: Conjugacy of rational relations is decidable

- Rational Relation → rational expression → sum of sumfree expressions.
- The union operation preserves conjugacy.
- Check the conjugacy of each sumfree expression by computing a common witness for it.
- The common witnesses of a sumfree expression can be computed in $\mathcal{O}(h \cdot m^3)$ where h is the star height of the expression and m is the length of the expression.

Conclusion

- A *set of pairs (or a relation) is conjugate* if each pair in the set is conjugate.

$\boxed{\text{Theorem: Conjugacy of rational relations is decidable}}$

- Extended abstract published in Developments in Language Theory (DLT) 2024.

- Application/ Motivation:

  Computing edit distance of finite state transducers (Aiswarya-Manuel-S. '2024).

Thank you